AMRMG
(Adaptive Mesh Refinement MultiGrid Code)

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Thanks to Dale Choi

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Motivation:

- AMR is needed to cover the short length scales of gravitational–wave sources ($\sim M$) and the long length scales of gravitational waves ($\sim 100M$).
- AMRMG will feed data into Hahndol, NASA Goddard's AMR evolution code.
- Goddard Numerical Relativity Group: John Baker, Joan Centrella, Dale Choi, Breno Imbiriba

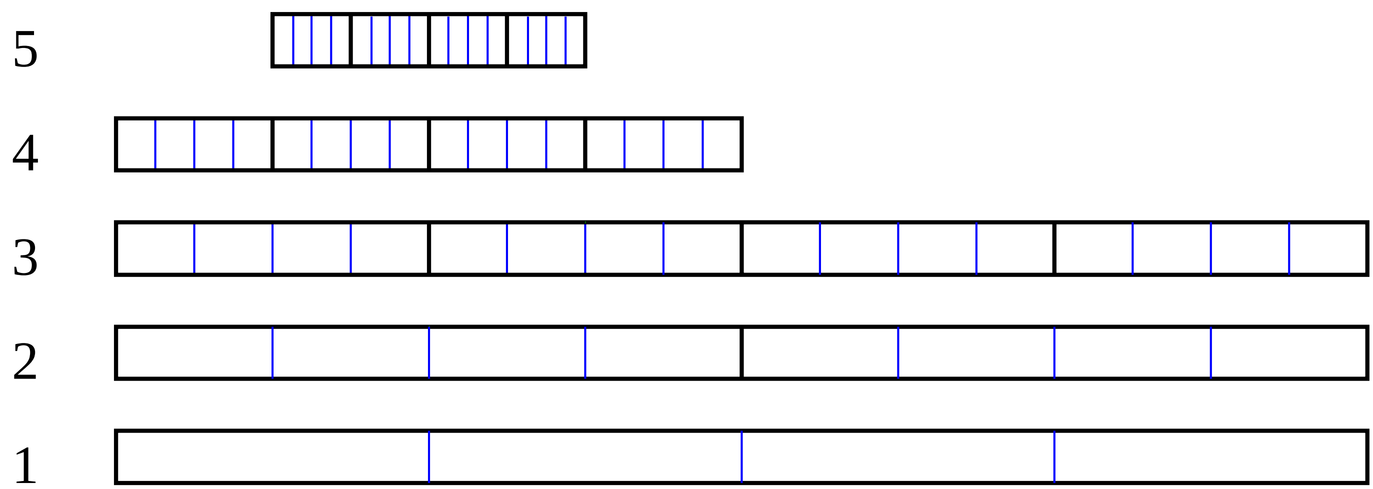
What does AMR mean for an elliptic solver?

- Grid resolution is determined locally as a part of the solution process.
- Specify the maximum desired truncation error, code constructs the grid to meet this requirement.

AMRMG uses PARAMESH

- Paramesh uses block refinement of the grid.
- Each block contains a fixed number of cells.
- Blocks are refined by bisection (1 block becomes 8 blocks in 3-D, 2 blocks in 1-D)
- AMRMG uses cell centered data.

1-D Example:



Multigrid Approach

- Solve the elliptic problem using a hierarchy of coarse and fine grids.
- The coarse grids are responsible for the long λ part of the solution.
- The fine grids are responsible for the short λ part of the solution.

- Approximate solutions at each refinement level are obtained by relaxation (smoothing).

  1-D Example:

$$\frac{\partial^2 \phi}{\partial x^2} = \rho \Longrightarrow \frac{\phi_{j+1} - 2\phi_j + \phi_{j-1}}{\Delta x^2} = \rho_j$$

$$\Longrightarrow \phi_j^{\mathrm{new}} = \frac{1}{2}(\phi_{j+1}^{\mathrm{old}} + \phi_{j-1}^{\mathrm{old}}) - \frac{\Delta x^2}{2}\rho_j$$

- Two choices for multigrid with AMR: relax across the entire computational domain or relax independently at each refinement level. AMRMG uses the second choice.

Nonlinear 2-Grid V-Cycle

Equation: $E(\,\cdot\,) = \rho$

$\{\,\}_2$ = fine grid, $\{\,\}_1$ = coarse grid

$\mathcal{R}$ = restriction $2 \to 1$, $\mathcal{P}$ = prolongation $1 \to 2$

(1) Solve $E_2(\,\cdot\,) = \rho_2$: Guess a solution $\tilde{\phi}_2 = 0$ and relax to obtain an approximate solution ϕ_2.

(2) Compute the coarse grid source:

$$\rho_1 = \mathcal{R}(\rho_2 - \underbrace{E_2(\phi_2))}_{\dagger} + \underbrace{E_1(\mathcal{R}\phi_2)}_{\ddagger}$$

- † subtract off the short λ part of the source that the fine grid got right
- ‡ add back the long λ part of the source that was removed by the short λ subtraction

(3) Solve $E_1(\,\cdot\,) = \rho_1$: Guess a solution $\tilde{\phi}_1 = \mathcal{R}\phi_2$ and relax to obtain an approximate solution ϕ_1 (or solve exactly for ϕ_1).

(4) Solve $E_2(\,\cdot\,) = \rho_2$: Guess a solution

$$\tilde{\phi}_2 = \phi_2 + \mathcal{P}(-\underbrace{\mathcal{R}\phi_2}_{\dagger} + \underbrace{\phi_1}_{\ddagger})$$

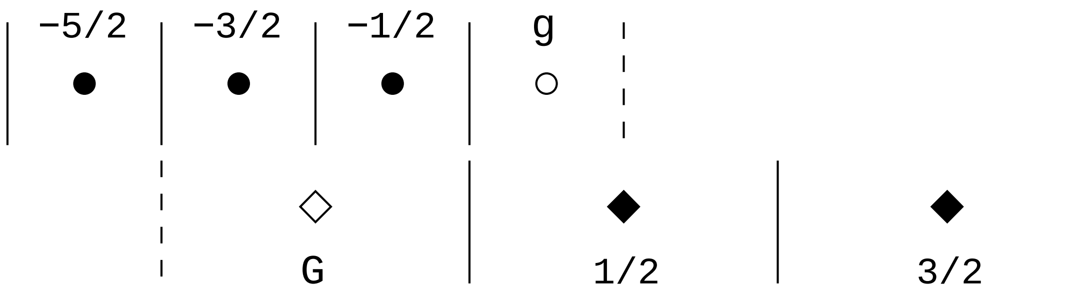
- † subtract off the long λ part of ϕ_2
- ‡ add ϕ_1, which contains primarily long λ information due to the construction of ρ_1

(5) Relax to obtain an improved approximate solution ϕ_2.

AMRMG is second order convergent

- Guard cells at coarse/fine interfaces must be filled to third order accuracy.

1-D Example:



$$\left.\frac{\partial^2 \phi}{\partial x^2}\right|_{-1/2} \Longrightarrow \frac{\phi_{-3/2} - 2\phi_{-1/2} + \phi_g}{\Delta x^2} + \mathcal{O}(\Delta x^2)$$

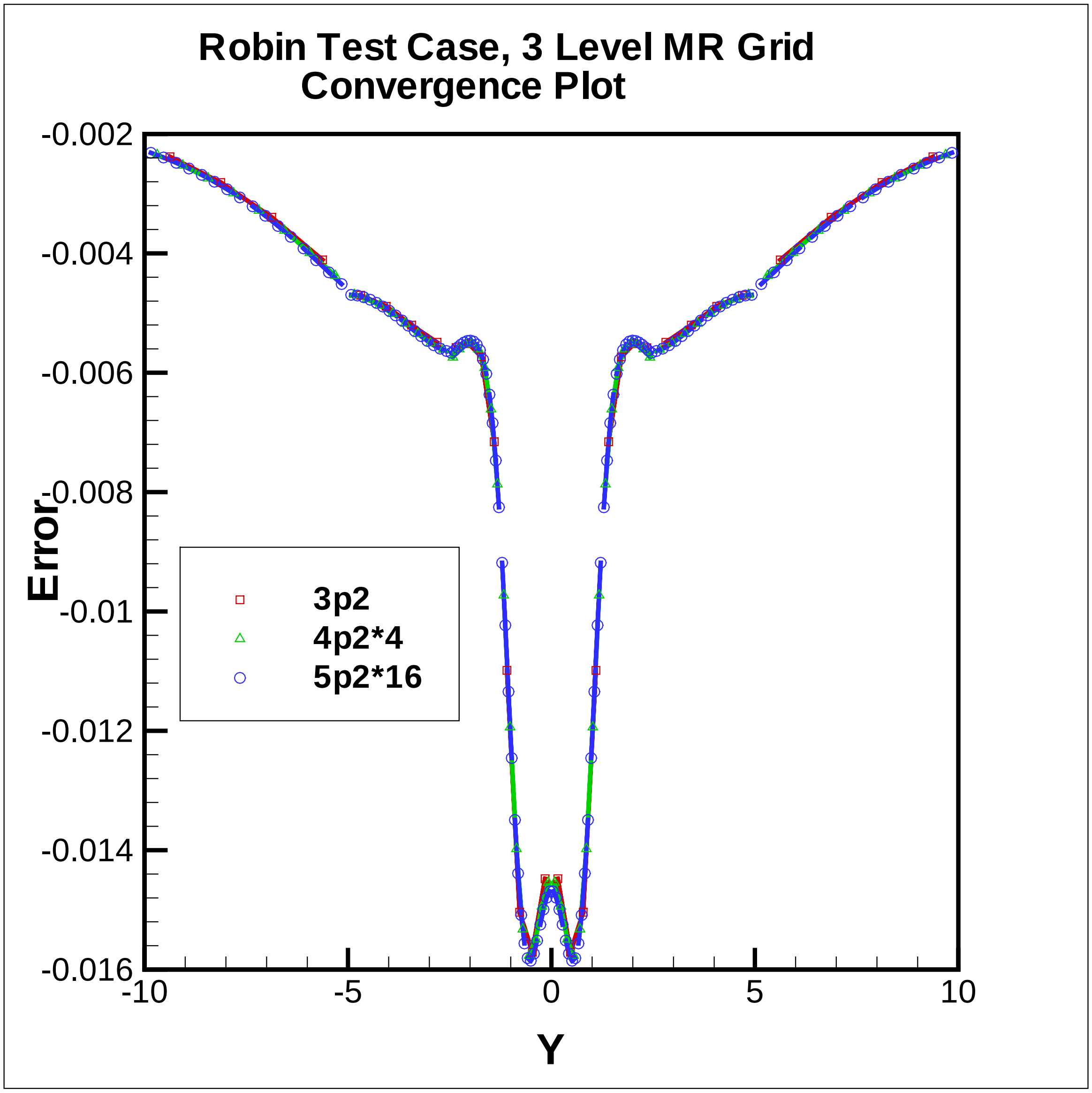
Guard cell ϕ_g must have errors no worse than $\mathcal{O}(\Delta x^3)$ or the overall second order convergence of the code will be spoiled.

- Restriction and prolongation must be handled carefully:

  $\mathcal{R}$, $\mathcal{P}$: linear interpolation ($\mathcal{O}(\Delta x^2)$ errors)
  $\mathcal{R}$, $\mathcal{P}$: cubic interpolation ($\mathcal{O}(\Delta x^4)$ errors)

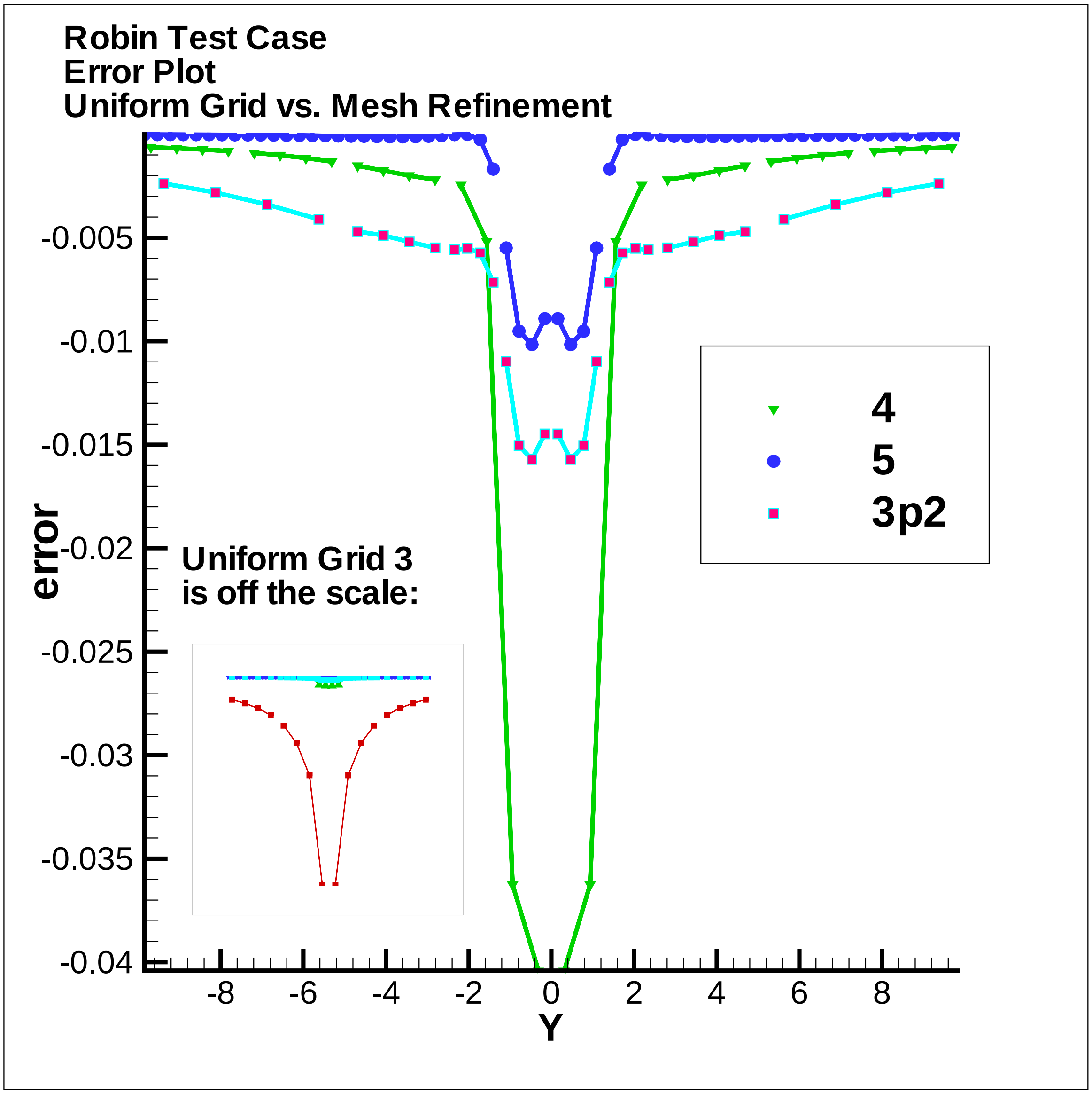
- Guard cell filling: Use $\mathcal{R}$ and $\mathcal{P}$
- Step 2: $\rho_1 = \mathcal{R}(\rho_2 - E_2(\phi_2)) + E_1(\mathcal{R}\phi_2)$
- Step 3: $\tilde{\phi}_1 = \mathcal{R}\phi_2$
- Step 4: $\tilde{\phi}_2 = \phi_2 + \mathcal{P}(-\mathcal{R}\phi_2 + \phi_1)$

FMR Convergence Test:



Error Comparison:

(Grid 4 $\sim 33\,$K pts, Grid 5 $\sim 262\,$K pts, Grid 3p2 $\sim 11\,$K pts)

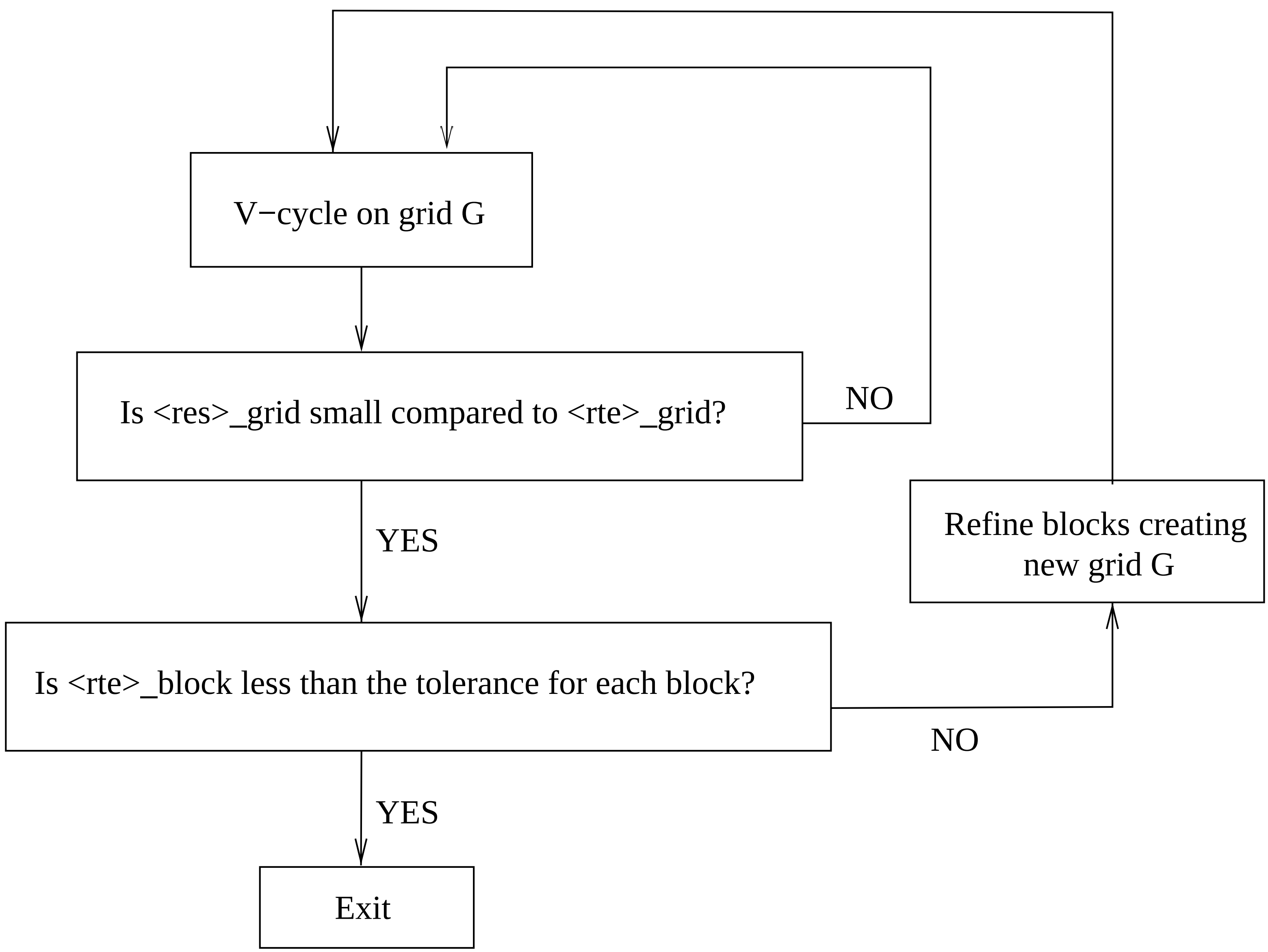


Error Control:

<res>_grid = norm of residual (E − rho) across the grid

<rte>_grid = norm of relative truncation error across the grid

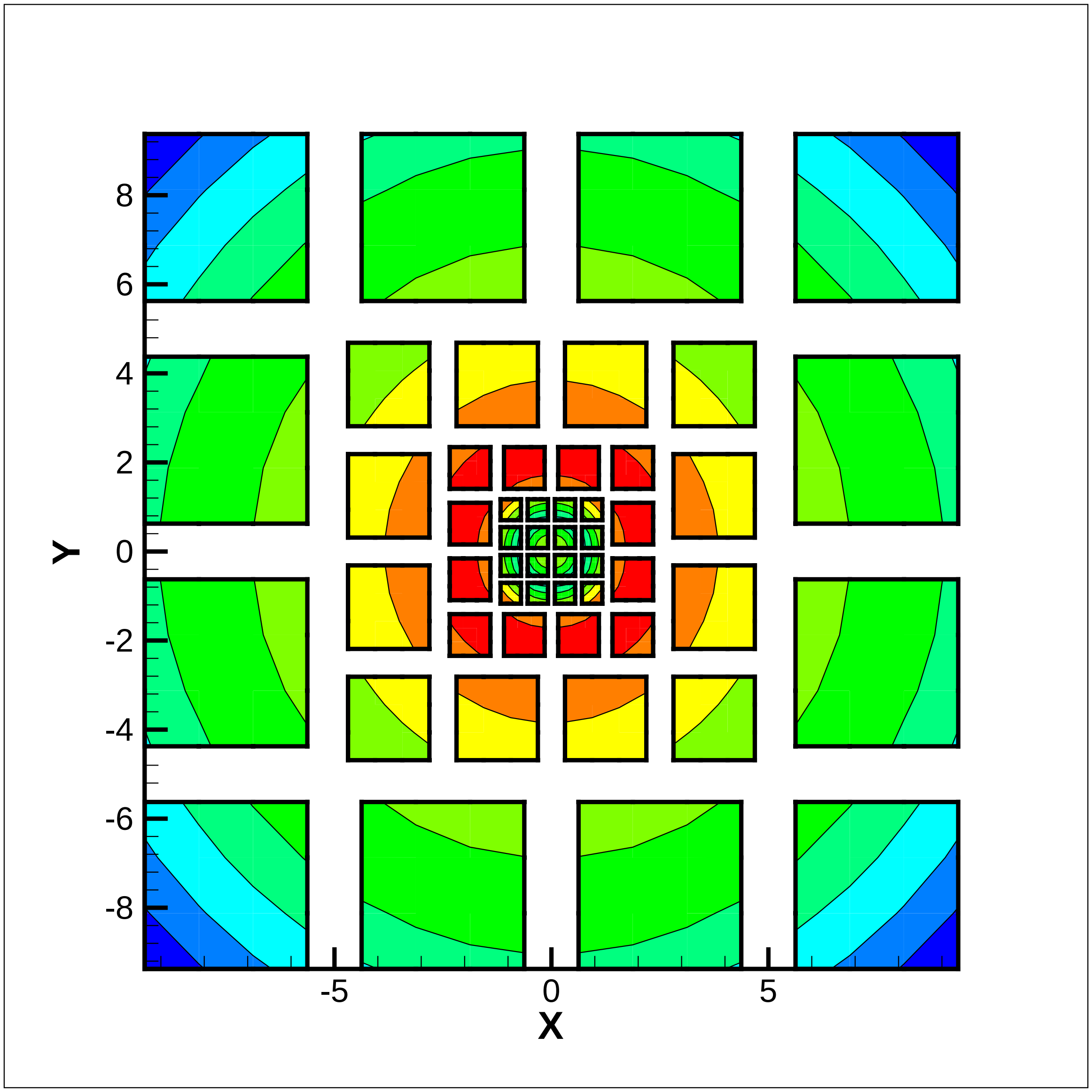
<rte>_block = norm of relative truncation error across a block



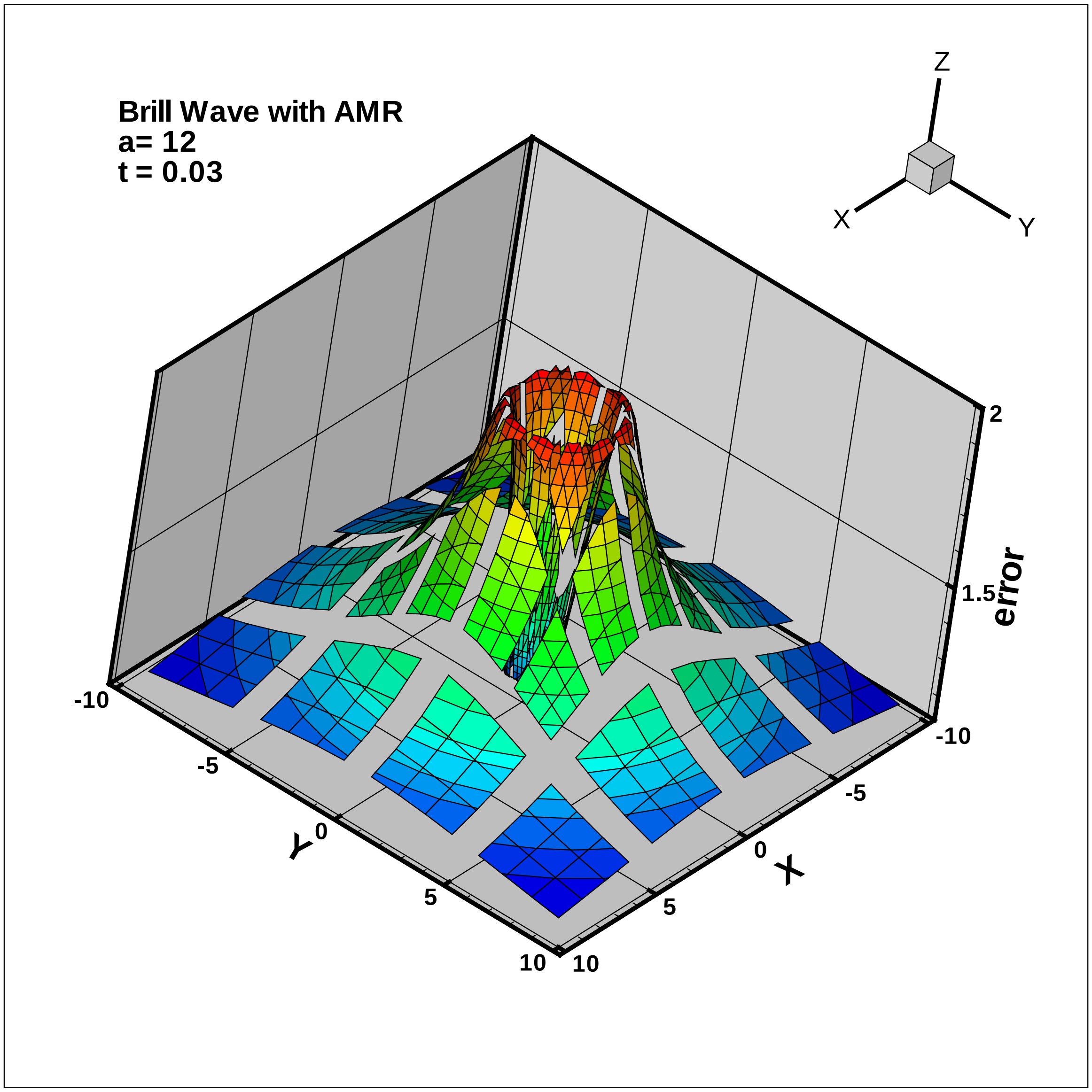
Brill Waves

- Metric:
$$ds^2 = \Psi^4 \left[e^{2q}(d\rho^2 + dz^2) + \rho^2 d\phi^2 \right]$$
- $q = a\rho^2 e^{-r^2}$ (Holz, Miller, Wakano, Wheeler)
- Extrinsic curvature: $K_{ij} = 0$
- Hamiltonian constraint:
$$\nabla^2 \Psi + \frac{1}{4}(q,_{\rho\rho} + q,_{zz})\Psi = 0$$
- Tolerance: $\langle$rte$\rangle$_block < 0.03

Brill Waves



Brill Waves



Brill Wave with AMR
a= 12
t = 0.03
Z
X
Y
2
1.5
error
-10
-5
0
5
10
Y
-10
-5
0
5
10
X